International Journal of Engineering & Technology Sciences
Volume 03, Issue 02, Pages 136-143, 2015

ISSN: 2289-4152

Free Vibration Analysis of Micro Beams Based on the Modified Couple Stress Theory, Using Approximate Methods

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Abstract

Keywords:

Assumed modes method,
Attached mass,
Free vibration,
Fundamental frequency,
Micro electro mechanical systems,
Modified couple stress theory.

Free vibration behaviour of micro scale Euler-Bernoulli cantilever beam carrying an attached mass is numerically investigated. This study is based on the modified couple stress elasticity theory. The non-dimensional frequency equation for the clamped- free micro beam is obtained using an approximate method for the first time. The method used in this study is the assumed modes method (Lagrange's equations). The method is used to exerted different numbers of admissible functions and their results are compared with each other. The influence of the attached mass at the free end upon the first dimensionless natural frequency is reported so that as the mass ratio (mass of the attached substance to the micro beam mass) is increasing, the dimensionless natural frequency is decreasing. Also the frequency shift for different values of the mass ratios is investigated. It is remarkable that the different behaviour of the micro beam is due to the mass inertia of the attached mass during oscillation. Comparison with the technical other article shows the expected frequency shift and accuracy of the solution procedure used.

Accepted: 26 March2015

1. Introduction

With the improvement of micro technology including concepts, mathematical modeling and fabrication, micro cantilever beams which their thickness is generally on the order of microns have been variously used in many applications of micro devices such as atomic force microscopes (AFMs) and different applications of micro-electro-mechanical systems (MEMS) [1-5]. Micro-electro-mechanical Systems or MEMS mainly consist of miniaturized mechanical and electro-mechanical elements such as devices and structures. In this technology the elements are made using micro fabrication techniques. It seems that the most practical and efficient elements of MEMS are micro sensors and micro actuators. As declared above the dimension of these structures are on the order of microns, on the other hand it is experimentally depicted that the size effect also plays a salient role in the mechanical behavior of micro structures [1-4]. Thus it is integral to consider the size dependency of micro structures along with their mathematical models [7]. Since controlled experiments in micro-scale are both difficult and expensive, the development of appropriate mathematical models for micro structures is an important issue concerning approximate analysis of these micro elements.

It is well-known that the classical theories of continuum mechanics do not include the size dependency of micro structures; as a result it is not decent to use the classical theories for the explanation and description of the mechanical behaviors of the micro structures under static and dynamic motivations. Thus, several continuum theories of the higher orders have been developed and used instead of the classical one. With the higher order theories the most difficult work especially in the couple stress theory, is to determine the micro structural material length scale parameter(s). The couple stress theory has three material parameters, for simplicity researchers introduced the modified couple stress theory (MCST). Beside of the two conventional equilibrium relationships in the classical couple stress related to force and momentum, they proposed an additional relation to constrain the couple phenomena. This relation considers the balance of moment of rotational momentum. This assumption makes the couple stress tensor symmetric [4, 7]. So the strain energy density function merely depends on the Cauchy strain tensor and the symmetric part of the curvature tensor. The effects of the dilatation gradient and the deviator stretch gradient are neglected.

Some researchers explored similar models for various investigations especially in relation to micro resonators and micro sensors. Fraga et al. [8] studied the mass detection using vertical and horizontal micro sensors and showed that the vertical design has more sensitivity but less accuracy. Dai et al. [2] studied the nano mechanical mass detection using nonlinear oscillators based on continuum elastic model and obtained that nonlinear oscillation leads to the unique resonant frequency shift due to mass adsorption, quite different from that in harmonic oscillation. Aghazadeh et al. [1] utilized the modified couple stress theory for the static and free vibration analysis of functionally graded micro scale beams. They stated the influences of material length scale parameter upon the first fundamental natural frequency when it is kept variable. Simsek [5] investigated the free vibration behavior of micro beams, placed on the non-linear foundation using the modified couple stress theory. He expressed the foundation parameters stiffness influences upon natural frequency. Rokni et al. [4] did research for the free vibration behavior of functionally graded micro polymer composite beams reinforced with MWCNTs based on the same modified theory at which the mechanical properties supposed to vary continuously through the axial generalized coordinate, they tried to describe the condition for which the maximum frequency can be obtained. With respect to the researches done, it is clear that the influences of the attached mass for pure metal structure upon the fundamental frequency of a clamped- free micro beam is not studied, although

Lee et al. [11] studied the similar case but they used the nonlocal elasticity theory. The solution procedure in this study is based on one of the most famous energy methods. It means that for the first time the free vibration analysis of micro beam carrying an attached mass, based on the modified couple stress theory along with the Euler-Bernoulli beam theories is studied without using the variation-like methods, like Hamilton's principle. Finally comparison with another technical article with respect to different numbers of admissible functions is done.

2. Mathematical formulation

2.1. Total Energies Based On the Modified Couple Stress Theory

Suppose a beam, plate or any other structure which is loaded (this load includes concentrated and distributed forces and moments). Because of the elastic property proportional to the modulus of toughness, the structure will bear the load. Before the final rupture this load induces deformation in the configuration of the structure and this deformation, in fact represents the stored strain energy. Now consider a flat, thin micro cantilever beam of length L, width b and thickness h. The cross section is assumed to be uniform along the length, Fig. 1. The strain energy for an elastic-linear, homogenous isotropic beam occupying the volume V is explained as follows:

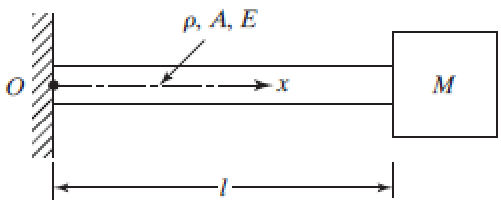

Fig. 1: micro-cantilever beam with an attached mass

$$U_s = \frac{1}{2}\int_V \left(\sigma_{ij}:\varepsilon_{ij} + m_{ij}:\chi_{ij}\right) dV \qquad (1)$$

The integrand function denotes the density function, where σ_{ij} designates the Cauchy stress tensor, ε_{ij} is the Cauchy strain tensor, m_{ij} represents the deviator part of the couple stress tensor and χ_{ij} stands for the symmetric curvature tensor. The tensors ε_{ij} and χ_{ij} are defined by Eq. (2, 3).

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{ij} + u_{ji}\right) \qquad (2)$$

$$\chi_{ij} = \frac{1}{2}\left(e_{ipq}\varepsilon_{qj,p} + e_{jpq}\varepsilon_{qi,p}\right) \qquad (3)$$

u In Eq. (2) is the displacement vector; e_{ipq} in Eq. (3) denotes the alternating tensor and comma stands for differentiation. Constitutive relations regarding the Cauchy stress tensor and the deviatoric part of the couple stress tensor is to be written in the following forms:

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk} \qquad (4)$$

$$m_{ij} = 2\mu l^2\chi_{ij} \qquad (5)$$

Where λ and μ are classic parameters. Also ν is the Poisson's ratio and E is the modulus of elasticity. By means of two constraint equations, the number of elastic parameters needed to describe the mechanical properties of a structure will reduce to two parameters, the constraint equations mentioned above are:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \ \mu = \frac{E}{2(1+\nu)} \qquad (6)$$

l In Eq. (5) is the material length scale parameter characterizing the effect of the couple stress theory. In the modified couple stress theory, the length scale parameter descripted by Mindlin [10], is also defined as a quantity whose square is equal to the ratio of the modulus of curvature to the modulus of rigidity. Therefore, introduction of the strain gradient into the formulation results in the definition of an intrinsic length scale parameter, which can be simply defined as a property of the elastic medium similar to modulus of elasticity, rigidity and Poisson's ratio.

The general displacement components in an Euler-Bernoulli beam can be represented by Benatta et al. [12]:

$$u_1(x_1, x_3, t) = -x_3 \frac{\partial w}{\partial x_1} \tag{7}$$

$$u_2(x_1, x_3, t) = 0 \tag{8}$$

$$u_3(x_1, x_3, t) = w(x_1, t) \tag{9}$$

In the above equations u_i , $(i = 1,2,3)$ are the general displacement components in x_1 , x_2 , x_3 directions.

Utilizing this displacement fields and Eq. (2)-(5), the elements of ε_{ij} , χ_{ij} , σ_{ij} and m_{ij} can be obtained as follows:

$$\varepsilon_{11} = -x_3 \frac{\partial}{\partial x_1}\left(\frac{\partial w}{\partial x_1}\right) \tag{10}$$

$$\chi_{12} = \chi_{21} = -\frac{1}{2}\frac{\partial}{\partial x_1}\left(\frac{\partial w}{\partial x_1}\right) \tag{11}$$

$$\sigma_{11} = -E x_3 \frac{\partial}{\partial x_1}\left(\frac{\partial w}{\partial x_1}\right) \tag{12}$$

$$m_{12} = m_{21} = -\mu l^2 \frac{\partial}{\partial x_1}\left(\frac{\partial w}{\partial x_1}\right) \tag{13}$$

The other components of the stresses and strains are zero. Substitution of the Eq. (10)-(13) into Eq. (1) results:

$$U_s = \frac{1}{2}\int_V \{E\left(\frac{\partial}{\partial x_1}\left(\frac{\partial w}{\partial x_1}\right)\right)^2 ({x_3}^2 + \frac{l^2}{2})\} \, dV \tag{14}$$

After calculating the potential energy, total kinetic energy of the Euler- Bernoulli beam carrying an attached mass can be expressed as:

$$T = \frac{1}{2}\int_0^L \rho A (\frac{\partial w}{\partial t})^2 \, dx_1 + \frac{1}{2} M (\frac{\partial w(L,t)}{\partial t})^2 \tag{15}$$

In which M represents the mass of the attached substance.

2.2. Approximate solution for free vibration

The new solution procedure used in this study is the assumed modes method-Lagrange's equations. This method obviates the need for exploiting the Hamilton's principle and other variation-like methods. This method is also different in approach from the other approximate methods like Rayleigh-Ritz method. The method consists of assuming a solution of the free vibration problem in the form of series composed of a linear combination of admissible functions like φ_i, which are functions of the spatial coordinates, multiplied by time-dependent generalized coordinates $q_i(t)$. So there is no need for validating and using the method of separation of variables. This method in essence treats a continuous system as a discrete system with n degrees of freedom.

$$w_n = \sum_{i=1}^{n} \varphi_i q_i(t) \tag{16}$$

Admissible functions should satisfy the geometric or natural boundary conditions. With the problem of cantilever based MEMS, the functions should be chosen such that the lateral deflection and slope at the joint place with the clamp should be zero. For the free end, the bending moment should be zero but due to the existence of the attached mass, an equilibrium equation for the shear forces should be considered. The admissible functions for different numbers of degree of freedom are as follows:

$$\varphi_1 = \left(\frac{x}{L}\right)^2 \tag{17}$$

$$\varphi_2 = \left(\frac{x}{L}\right)^3 \tag{18}$$

$$\varphi_3 = \left(\frac{x}{L}\right)^4 \tag{19}$$

$$\varphi_4 = \left(\frac{x}{L}\right)^5 \quad (20)$$

$$\varphi_4 = \left(\frac{x}{L}\right)^6 \quad (20)$$

The principal equation known as Lagrange equation based on the assumed modes method for conservative systems is defined as:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial U_s}{\partial q_r} = 0, \quad r = 1,2,\dots,n, \quad (21)$$

$$q(t) = e^{i\omega t} \quad (22)$$

In Eq. (22), ω is the fundamental frequency and t shows the time variable.

By considering the harmonic dependence of the generalized coordinate and after some mathematical operations using MATLAB software a homogenous system of equations is obtained. Due to the linear algebra it is expected that the determinant of the coefficient matrix should be equal to zero; which gives the frequency equation. According to the number of admissible functions chosen, the corresponding fundamental frequencies are calculated. Dimensionless fundamental frequency is defined as:

$$\beta = \omega L^2 \sqrt{\frac{\rho A}{EI + GAl^2}} \quad (23)$$

In which ρ, A, E, I, G are density, cross sectional area, modulus of elasticity, second moment of inertia about the x_2 axis and modulus of rigidity.

In all tables:

(a) Is related to three admissible functions.
(b) Is related to four admissible functions.
(c) Is related to five admissible functions.

Table 1: Convergence and Comparison of dimensionless fundamental natural frequency ($R = 0$)

$\frac{l}{h}$	(a) $n = 3$	(b) $n = 4$	(c) $n = 5$	Ref. Rokni et al. (2014)
0.2	3.9165	3.91523	3.9153	-
0.41	4.9846	4.9831	4.9831	4.9724
0.6	6.2503	6.2502	6.2502	-
0.8	7.7376	7.7352	7.7352	-

Table 2: Convergence of dimensionless fundamental natural frequency ($R = 0.01$)

$\frac{l}{h}$	(a) $n = 3$	(b) $n = 4$	(c) $n = 5$
0.2	3.8402	3.8392	3.8392
0.41	4.8875	4.8862	4.8863
0.6	6.1304	6.1287	6.1288
0.8	7.5869	7.5849	7.5849

Table 3: Convergence of dimensionless fundamental natural frequency ($R = 0.1$)

$\frac{l}{h}$	(a) $n = 3$	(b) $n = 4$	(c) $n = 5$
0.2	3.3054	3.3048	3.3048
0.41	4.2068	4.2062	4.2061
0.6	5.2766	5.2757	5.2757
0.8	6.5303	6.5293	6.5293

Table 4: Convergence of dimensionless fundamental natural frequency ($R = 1$)

$\frac{l}{h}$	(a) $n = 3$	(b) $n = 4$	(c) $n = 5$
0.2	1.7341	1.7341	1.7341
0.41	2.2071	2.2071	2.2071
0.6	2.7683	2.7683	2.7683
0.8	3.4260	3.4261	3.4261

Table 5: Convergence of dimensionless fundamental natural frequency ($R = 10$)

$\frac{l}{h}$	(a) $n = 3$	(b) $n = 4$	(c) $n = 5$
0.2	0.6029	0.6029	0.6029
0.41	0.7673	0.7673	0.7673
0.6	0.9624	0.9624	0.9624
0.8	1.1910	1.1911	1.1911

Table 6**:** Frequency shift due to the attached mass. Using four admissible functions. $\left(\frac{l}{h}\right) = 0.41$

R	β	Ref. Rokni et al.	Δf(%100)
0.01	4.8862	4.9724	1.73
0.1	4.2062	4.9724	15.41
1	2.2071	4.9724	55.61
10	0.7673	4.9724	84.57

With Table 6, Δf shows the frequency shift. Also the quantities belonging to the reference are calculated with no attached mass boundary conditions.

3. Results and discussion

Based on the modified couple stress and Euler-Bernoulli beam theories the dimensionless fundamental frequency and frequency shift of a micro cantilever beam carrying an attached mass at the free end was studied by using an approximate method. This new solution procedure does not need variation-wise operations, instead the terms of kinetic and potential energies should be used. The dimensionless fundamental frequency for different values of ratio of the material length scale parameter to the micro beam thickness and different values of the mass ratio are calculated. Also the mentioned procedure is accompanied by three types of admissible functions. The differences and also the convergence of the answers are clear as the number of admissible functions is increasing.

4. Conclusions

The following conclusions can be drawn from this research:

a. The effect of the attached mass upon the fundamental frequency is increasing when the mass ratio becomes greater in quantity.

b. As the mass ratio is increasing, the larger frequency shift is observed.

c. As the mass ratio is increasing, the frequency shift is going to be negative.

d. By using an added mass, a kind of structural damping is heightened.

e. Assumed modes method is an appropriate one, so it can be used for future studies.

f. For the fundamental frequency, using four admissible functions will give accurate results.

g. For the modified couple stress theory, natural frequencies depend on both bending stiffness and the stiffness related to the scale parameter.

References

[1] R Aghazadeh, E Cigeroglu, S Dag. Static and free vibration analyses of small-scale functionally graded beams possessing a variable length scale parameter using different beam theories. *European Journal of Mechanics-A/Solids* 2014; 46: 1-11. 2014.01.002

[2] Dai MD, Eom K, Kim CW. Nanomechanical mass detection using nonlinear oscillations. *Applied Physics Letters* 2009; 95(20): 203104.

[3] Ke LL, Yang J, Kitipornchai S. Nonlinear free vibration of functionally graded carbon nanotube-reinforced composite beams. *Composite Structures* 2010; 92(3): 676-683.

[4] H Rokni, A Milani, RJ Seethaler. Size-dependent vibration behavior of functionally graded CNT-Reinforced polymer microcantilevers: Modeling and optimization. *European Journal of Mechanics-A/Solids* 2015; 49: 26-34.

[5] Simsek M. Nonlinear static and free vibration analysis of microbeams based on the nonlinear elastic foundation using modified couple stress theory and He's variational method. *Composite Structures* 2014; 112: 264-272.

[6] Yang F, Chong ACM, Lam DCC et al. Couple stress based strain gradient theory for elasticity.Int. J. *Solids Structures* 2002; 39(10): 2731-2743.

[7] M Fathalilu, M Sadeghi, G Rezazadeh. Micro-inertia effects on the dynamic characteristics of micro-beams considering the couple stress theory. *Mechanics Research Communications* 2014; 6413(14): 00084-6

[8] Sánchez-Fraga R, Ponce-Ponce VH, Mendoza-Acevedo S, Villa-Vargas LA, Estrada-Vázquez H. Design and simulation of mass sensors based on horizontally actuated silicon cantilevers. *Microelectronic Engineering* 2014; 119: 83-88.

[9] Toupin RA. Elastic materials with couple stresses. *Archives of Rational for Mechanical and Analysis* 1965; 11: 385–414.

[10] Mindlin RD, Tiersten HF. Effects of couple-stresses in linear elasticity, *Archive for Rational Mechanics and Analysis* 1962; 11: 415–448.

[11] Lee HL, Hsu JC, Chang WJ. Frequency shift of carbon-nanotube-based mass sensor using nonlocal elasticity theory. *Nanoscale research letters* 2010; 5(11): 1774-1778.

[12] Benatta MA, Mechab I, Tounsi A, Adda EA. Static analysis of functionally graded short beams including warping and shear deformation effects. *Computational Materials Science* 2008; 44(2): 765-773.

[13] Ginsberg JH. Mechanical and Structural Vibrations: Theory and Applications. *John Wiley and Sons* 2001.

[14] Rao SS. Vibration of continuous systems. *John Wiley and Sons* 2007.