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A New Method for the Analysis of Beam's Deflection Using the Taylor Series Expansion

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Abstract

Deflection and rotation of beams can be determined by various methods available in the literature. In this investigation, a new method based on the Taylor series expansion called "Taylor method" is introduced. The method yields two single equations for calculation of deflection and the rotation of the beam regardless of the loading and supports conditions. The method employs the governing differential equations for deflection of beams in mechanics of solids along with Taylor series expansion. The equations of shear force and bending moment can be determined using Taylor series and without resorting to their corresponding diagrams. The method is examined by solving a number of beam problems. The results indicate that the method can be used with confidence for solving any problemrelated to the beam's deflection. The method also seems to be more convenient and more efficient than the other methods.

1. Introduction

Deflection and rotation ofbeams can be determined by various approaches some of which are direct integration method, moment-area approach, energy method which includes Castigliano's theorems and unit load technique, stiffness and flexibility methods, superposition principal, three bending moments technique and singularity functions method [1-3].

Some of the methods such as integration and singularity functions lead to equations which describe deflection and rotation throughout the length of the beam. However, these methods are sometimes cumbersome for the beams with multiple supports, spans and combined loadings. The other approaches such as moment-area, flexibility, stiffness, energy and three moment techniques can predict the deflection and rotation only at specified points. Moreover, except for the singularity functions, deflection and rotation are defined by various equations for beams with multiple supports and combined loadings. Also, Singularity functions are not very popular as they need symbolic mathematical expressions for discontinuous functions of load, shear and moment along a beam.

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In this investigation, a new method based on the Taylor series expansion is developed for the evaluation of beam's deflection and rotation. The method utilizes the elementary equations found normally in the books of solid mechanics. The validity of the method is also examined by solving a number of problems.

2. Taylor Series Expansion

The Taylor series expansion defines a function in the vicinity of a point in terms of the values of the function and its derivatives at that point. The Taylor series expansion of the function $y(x)$ at the point $x=a$ can be expressed as

$$y(x) = y(a) + \frac{(x-a)y^{(1)}(a)}{1!} + \frac{(x-a)^2 y^{(2)}(a)}{2!} + \frac{(x-a)^3 y^{(3)}(a)}{3!} + \cdots + \frac{(x-a)^n y^{(n)}(a)}{n!} \tag{1}$$

in which $y^{(n)}(a)$ is the *n*thderivative of the function *y(x)* at $x=a$. The expansion at point $x=0$, known as Maclaurin expansion series, becomes

$$y(x) = y(0) + \frac{x y^{(1)}(0)}{1!} + \frac{x^2 y^{(2)}(0)}{2!} + \frac{x^3 y^{(3)}(0)}{3!} + \cdots + \frac{x^n y^{(n)}(0)}{n!} \tag{2}$$

Taylor and Maclaurin expansions are widely used in solving differential equations and for calculation of the function at a specified point in terms of a finite number of its derivatives.

3. Equations Governing the Beam's Deflection

If the function $y(x)$ in the Taylor series is assumed to be the deflection of the beam, then according to the governing differential equations for the deflection of the beams, the first, second, third and the fourth derivatives of the function are corresponding to the rotation, bending moment, shear force and the uniformly distributed load, respectively. These are defined by the following relations

$$\theta(x) = \frac{dy(x)}{dx} = y^{(1)}(x) \tag{3}$$

$$M(x) = EIy^{(2)}(x) \tag{4}$$

$$V(x) = EIy^{(3)}(x) \tag{5}$$

$$q(x) = EIy^{(4)}(x) \tag{6}$$

Eqs. (4) to (6) are derived in accordance with the sign conventions for positive bending moment, shear force and the distributed load as shown in Figure 1.

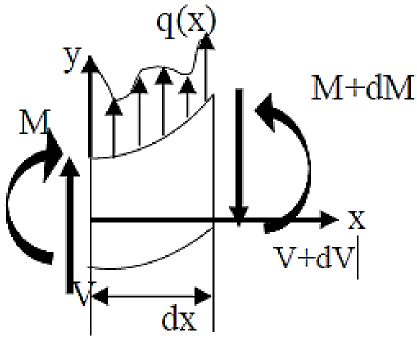

Figure 1. The sign conventions used for the positive bending moment, shear force and distributed load

In Eqs. (3) to (6), $q(x)$, $M(x)$, $V(x)$ and $\theta(x)$ denote the distributed load, bending moment, shear force and rotation of the beam at any arbitrary section of the beam, respectively. The signs of the loads must conform to the Cartesian coordinate system as illustrated in Figure 1. Substituting Eqs. (3) to (6) into Eqs. (1) and (2), results in

$$y(x) = y(a) + \frac{(x-a)\theta(a)}{1!} + \frac{(x-a)^2 M(a)}{2!EI} + \frac{(x-a)^3 V(a)}{3!EI} + \frac{(x-a)^4 q(a)}{4!EI} + \frac{(x-a)^5 q\prime(a)}{5!EI} + \frac{(x-a)^6 q\prime\prime(a)}{6!EI} + \cdots \quad (7)$$

$$y(x) = y(0) + \frac{x\theta(0)}{1!} + \frac{x^2 M(0)}{2!EI} + \frac{x^3 V(0)}{3!EI} + \frac{x^4 q(0)}{4!EI} + \frac{x^5 q\prime(0)}{5!EI} + \frac{x^6 q\prime\prime(0)}{6!EI} + \cdots \quad (8)$$

Eq. (8) is usually referred to as the elastic curve of the beam. Here, prime denotes the derivative with respect to x. If the left end of the beam is selected as the origin of the x axis, then the value of $q(0)$, $M(0)$, $V(0)$ and $\theta(0)$ will denote the distributed load, bending moment, shear force and the rotation of the beam at the left end, respectively. It is evident that the origin of the coordinate system can be set at any arbitrary point. However, the left end of the beam can be more convenient and will simplify the computations. It is interesting to note that the number of terms in Taylor series given by Eqs. (7) and (8) is finite. For instance, in the case of uniformly distributed loads, the derivatives $q'(0)$, $q''(0)$,.... are zero, while for a linear distributed load, $q''(0)$, ... are zero.

Another feature of the method described in this paper is that the rotation, bending moment and shear force at any section of the beam can be obtained using the Taylor series expansion (Eq.(2)) along with Eqs. (3) to (6). This will enable us to determine the distribution of bending moment and shear force in the beam without resorting to sectioning technique and integration which are sometimes, in particular for complex loadings and multiple supports, cumbersome. From Eq. (8) and using Eqs. (3) to (5), the following relations are derived

$$\theta(x) = \theta(0) + \frac{xM(0)}{EI} + \frac{x^2 V(0)}{2!EI} + \frac{x^3 q(0)}{3!EI} + \frac{x^4 q\prime(0)}{4!EI} + \cdots \quad (9)$$

$$M(x) = M(0) + xV(0) + \frac{x^2 q(0)}{2!} + \frac{x^3 q'(0)}{3!} + \frac{x^4 q''(0)}{4!} + \cdots \qquad (10)$$

$$V(x) = V(0) + xq(0) + \frac{x^2 q'(0)}{2!} + \frac{x^3 q''(0)}{3!} + \frac{x^4 q^{(3)}(0)}{4!} + \cdots \qquad (11)$$

4. Method Validation

In this section, the validity of the method is examined by solving a number of statistically determinate and indeterminate beams of different loading conditions and supports.

4.1. Statistically Determinate Problems

4.1.1. A Cantilever Under Distributed and the Point Load

A cantilever under a distributed load is illustrated in Figure 2. With regard to the fact that $\theta(0)$ and $y(0)$ are zero at the left support, Eq. (8) for deflection of the beam becomes

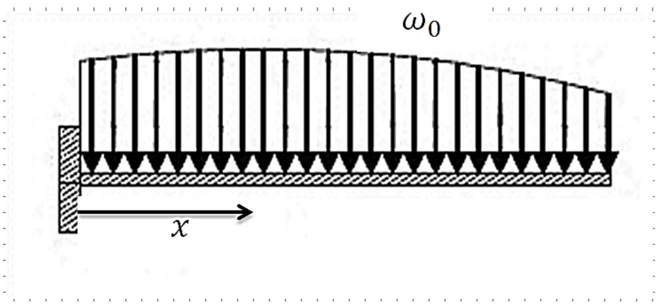

Figure 2.A cantilever under a distributed load

$$y(x) = \frac{x^2 M(0)}{2!EI} + \frac{x^3 V(0)}{3!EI} + \frac{x^4 q(0)}{4!EI} + \frac{x^5 q'(0)}{5!EI} + \frac{x^6 q''(0)}{6!EI} + \cdots \qquad (12)$$

This equation describes the deflection along the beam regardless of the type of loading which can be concentrated force, distributed or bending moment. For a uniformly distributed loadwe can write

$$y(x) = \frac{x^2 M(0)}{2!EI} + \frac{x^3 V(0)}{3!EI} + \frac{x^4 q(0)}{4!EI} \qquad (13)$$

From equilibrium equations, the reactions at the support of the beam are determined as

$$V(0) = \omega_0 L \ , \ \ M(0) = -\frac{\omega_0 L^2}{2} \ , \ \ q(0) = -\omega_0 \qquad (14)$$

Substituting the above values into Eq. (13), the equation of the elastic curve of the beam and the maximum deflection are derived as

$$y(x) = \frac{-x^2\omega_0 L^2}{4EI} + \frac{x^3\omega_0 L}{6EI} - \frac{x^4\omega_0}{24EI} \quad (15)$$

$$y(x) = \frac{\omega_0 x^2}{24EI}(-6L^2 + 4xL - x^2) \quad (16)$$

$$y(x = L) = y_{Max} = \frac{-\omega_0 L^4}{8EI} \quad (17)$$

For a concentrated load applied at the free end of the beam as depicted in Figure 3, Eq. (13) reduces even more as follows

$$y(x) = \frac{x^2 M(0)}{2EI} + \frac{x^3 V(0)}{6EI} \quad (18)$$

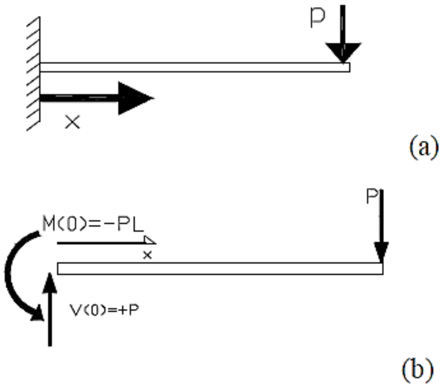

Figure 3. (a) A cantilever under a concentrated load and (b) the free body diagram of the beam

In the above equation, $V(0)$ and $M(0)$ are the shear force and bending moment, respectively, which can be determined from force and moment equilibrium equations given as below

$$V(0) = P, M(0) = -PL \quad (19)$$

Substituting these values into Eq. (18), equation of the elastic curve of the beam and the maximum deflection are determined as follows

$$y(x) = -\frac{PLx^2}{2EI} + \frac{Px^3}{6EI} \quad (20)$$

$$y(x) = -\frac{Px^2}{6EI}(3L - x) \quad (21)$$

$$y_{max} = y(x = L) = -PL^3/3EI \quad (22)$$

4.1.2. A Simply Supported Beam

A simply supported beam under a distributed load is shown in Figure 4. For this beam, we have $y(0) = 0, M(0) = 0$ and $M(L) = 0$. Therefore, from the Taylor series expansion given by Eq. (8), the deflection of the beam can be expressed as

$$y(x) = x\theta(0) + \frac{x^3 V(0)}{3!EI} + \frac{x^4 q(0)}{4!EI} + \frac{x^5 q'(0)}{5!EI} + \frac{x^6 q''(0)}{6!EI} + \cdots \quad (23)$$

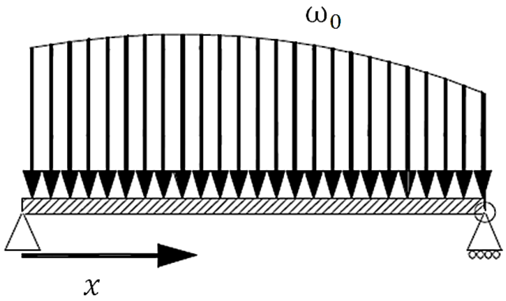

Figure 4. A simply supported beam under a distributed load

For a simply supported beam under a point load applied at the center of the beam, we have $q'(0) = q''(0) = \cdots = 0$. So, Eq. (23) will be reduced to

$$y(x) = x\theta(0) + \frac{x^3 V(0)}{6EI} \quad L/2 \leq x \leq 0 \tag{24}$$

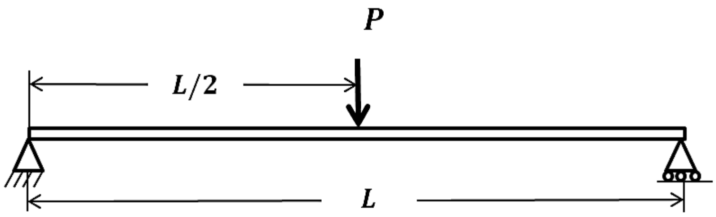

Figure 5. A simply supported beam under a point load applied at its center

The value of the shear force at the left support is obtained from equilibrium equation as $V(0) = P/2$. By differentiating Eq. (24) with respect to *x*, the rotation of the beam can be determined as

$$\theta(x) = \theta(0) + \frac{Px^2}{4EI} \tag{25}$$

Since, the value of the rotation is zero at the point of load application, the value of the rotation at the left support can be obtained by using Eq. (25) as

$$\theta\left(x = \frac{L}{2}\right) = \theta(0) + \frac{PL^2}{16EI} = 0 \;\Rightarrow\; \theta(0) = -\frac{PL^2}{16EI} \tag{26}$$

Substituting the value of $\theta(0)$and $V(0)$ into Eq. (25), equation of the elastic curve of the beam and the maximum deflection are determined as

$$y(x) = \frac{-Px}{48EI}(3L^2 - 4x^2) L/2 \leq x \leq 0 \tag{27}$$

It should be noted that Eq. (27) holds only for the region of the beam located on the left hand side of the load, , $0 \leq x \leq L$. If x is replaced by $L - x$, then the equation of deflection for the right hand side of the load is obtained.

4.1.3. A cantilever under linear distributed load

A cantilever carries a linear distributed load as shown in Figure 6. Taylor series for the deflection of the beam at $x = 0$ can be written as

$$y(x) = y(0) + x\theta(0) + \frac{x^3 V(0)}{3!EI} + \frac{x^4 q(0)}{4!EI} + \frac{x^5 q\prime(0)}{5!EI} \tag{28}$$

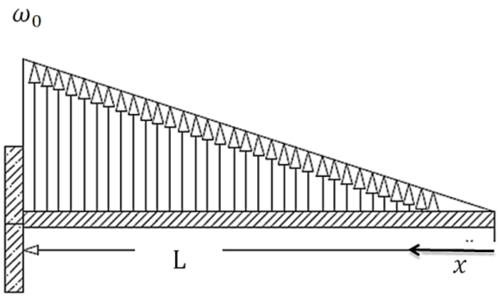

Figure 6. A cantilever under linear distributed load

The shear force and bending moment at the free end and also the distributed load and its derivatives at any point can be written as

$$V(0) = 0,\ \ M(0) = 0,\ \ q(x) = \omega_0 \left(\frac{x}{L}\right), \quad q'(x) = \frac{\omega_0}{L} \tag{29}$$

Substituting Eq. (29) into Eq. (28), we arrive at

$$y(x) = y(0) + x\theta(0) + \frac{x^5 \omega_0}{120EIL} \tag{30}$$

$$\theta(x) = \theta(0) + \frac{x^4 \omega_0}{24EIL} \tag{31}$$

Using the boundary conditions $\theta(L) = 0$ and $y\ (L)\ = 0$, the values of deflection and rotation at $x = 0$ can be computed from Eqs. (30) and (31) as follows

$$\theta(L) = 0 \Rightarrow \ \theta(0) = -\frac{\omega_0 L^3}{24EI} \tag{32}$$

$$y(L) = 0 \Rightarrow y(0) = \frac{\omega_0 L^4}{30EI} \tag{33}$$

Substituting the above relations into Eq. (22), the relation for deflection of the beam can be obtained as

$$y(x) = \frac{\omega_0}{120EI}(4L^2 - 5xL^4 + x^5) \tag{34}$$

5. Conclusions

The new method developed in this work for bending analysis of beamswas validated by a number of relative problems which are of more interesting in the mechanics of solids. The authors of this work would like to call this approach as "Taylor method". The method has the following advantages:

a) Similar to integration and singularity functions methods and unlike the methods such as energy and moment area, Taylor method provides general equations for describing deflection and the rotation of the beam.

b) Fewer computations and mathematical operations are required for deriving the deflection and the rotation equations of the beam, as it is needed in integration method.

c) Beams of multiple spans, supports and complex loading can be analyzed by this method.

d) The method enables the user to determine the equations of shear force and bending moment without resorting to their corresponding diagrams.

e) The beam equations can be easily obtained in each segment of the beam by writing Taylor series expansion at the beginning of that segment. On the whole, Taylor method enjoys nearly the advantages of all methods available in the literature on one hand and lacks the shortcomings of those methods on the other hand. Therefore, to the best of author's belief, Taylor method can be regarded as one of the most comprehensive methods for beam analysis.

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